Grade 11/12 Math Circles
November 8, 2023
P-adic numbers, Part 2 - Solutions

Exercise Solutions

Exercise 1

Find solution for y if $x = \frac{1}{2}$. Is it a rational number?

Exercise 1 Solution

Let $x = \frac{1}{2}$, then $y^2 = (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^8 = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} = \frac{81}{256}$, therefore $y = \sqrt{\frac{81}{256}} = \frac{9}{16}$ is indeed rational.

Exercise 2

Check that $x_1 = 1$ is a solution to $3 + 15x_1 =_9 0$.

Exercise 2 Solution

$3 + 15x_1 = 3 + 15 = 18$. The remainder of 18 after division by 9 is 0, so the equality holds.

Exercise 3

What's bigger the distance between $s_1 = \ldots 010101$ and $s_2 = \ldots 101010$ or the absolute value of their sum in 3-adics?

Exercise 3 Solution

$$s_2 - s_1 = \ldots 101010 + \ldots 212122 = \ldots 020202,$$

$$s_2 + s_1 = \ldots 111111,$$

so

$$|s_1 - s_2|_3 = |\ldots 020202|_3 = 0$$

$$|s_1 + s_2|_3 = |\ldots 111111|_3 = 0.$$

We observe that the distance is the same as the absolute value of the sum.

Problem Set Solutions

1. Find all solutions to $x^2 = 1$ *mod* 11 and *mod* 13.

   Solution: There are two solutions to $x^2 = 1$ *mod* 11, namely 1 and $-1 =_{11} 10$:

   $$1^2 =_{11} 1, \quad (-1)^2 =_{11}= (10^2) =_{11} 100 =_{11} 1$$

   The same thing goes for *mod* 13.

2. For $p = 3$, $x = 36$, we have that $x = 1100_3$. Find $|x^2|_p$.

   Solution:
   Start by finding $36^2 = 1296$ in base 10. Then we calculate $1296 = 1210000_3$ in base 3 by dividing it by powers of 3 starting with 3^6.
   Next we compute $|x^2|_3 = |1210000|_3 = 3^{-5}$.

3. Show that if x is a rational number then the product of all the numbers $|x|_p$ for p a prime is 1.

   Solution: Expanding a rational number x by prime factors

   $$x = \pm p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n},$$

   where p_j are different prime numbers, a_j are integers, and using definition of p-adic absolute value, we obtain

$$|x|_{p_j} = p_j^{-a_j}, \quad |x|_p = 1, p \neq p_j, \quad |x|_\infty = p_1^{-a_1} p_2^{-a_2} \dots p_n^{-a_n}.$$

These facts imply our statement.

4. Show that $d(x,z) = |x-z|_p \leq d(x,y) + d(y,z)$ for integer p-adic numbers. Better still, show that $d(x,z) \leq \max d(x,y), d(y,z)$.

Solution: Let $z = \pm \sum_j a_j p^j = a_0 p^0 + a_1 p^1 + \dots$, and let k be the first index such that $a_k \neq 0$.
Keeping that in mind. If k is the highest power that divides xy and j is the highest power that divides yz. If $k \neq j$ and $min(k,j) < m \leq \max(k,j)$, then p^m will divide one of xy or yz but not the other so $p^m(xy) + (yz)| = xz$.
In that case the highest power that divides xz is $\min(k,j)$ and $|xz| = p^{-\min(k,j)} = \max(p^k, p^j) = \max(|xy|_p, |yz|_p)$.
If on the other hand if $k = j$ then the maximum power that divides $(xy) + (yz) = xz$ is $k = j$ and $|xz| \leq |xy|_p = |yz|_p$ so

$$d(x,z) \leq \max(d(x,y), d(y,z))$$

and thus

$$d(x,z) \leq \max(d(x,y), d(y,z)) + \min(d(x,y), d(y,z)) = d(x,y) + d(y,z).$$